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$\hbar \to 0$ and classical-quantum correspondence in the kicked Harper model

Indubala I. Satija[1] and Tomaž Prosen[2]
[1]*Department of Physics, George Mason University, Fairfax, Virginia 22030*
[2]*Physics Department, Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, SI-1111 Ljubljana, Slovenia*
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We investigate classical-quantum correspondence for the kicked Harper model for extremely small values of the Planck constant $\hbar$. In the asymmetric case a pure quantum state shows a clear signature of classical diffusive as well as superdiffusive transitions asymptotically independent of $\hbar$. However, for the symmetric case, the $\hbar$ independent behavior occurs only for the renormalized parameter $\bar{K}=K/(2\hbar)$ with intriguing features such as a sharp transition from integrable to nonintegrable transport at $\bar{K}=\pi/2$, a series of transitions at multiples of π, and the periodicity of the transmission probability. We suggest that even as $\hbar\to 0$, the quantum dynamics is influenced by cantori and additional features emerge in their absence.

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Localized transport of a quantum system in the regime where the corresponding classical system exhibits deterministic diffusive behavior is one of the most surprising aspects of nonintegrable Hamiltonian systems [1]. However, the correspondence principle requires some signatures of various classical transitions such as the breakup of Kolmogorov, Arnold, and Moser (KAM) tori leading to diffusive transport and the emergence of *accelerator modes* (AMs) resulting in superdiffusive anomalous transport. In this paper, we describe quantum signatures of various classical transitions in transport characteristics emphasizing the crossover effects from large to small values of the effective Planck's constant $\hbar$. As $\hbar\to 0$, the quantum system exhibiting localization in one of the phase-space directions is found to *feel* the effects of all classical transitions. However, in the absence of localized transport, the quantum system exhibits many surprising features and appears to be insensitive to the classical dynamics.

The problem of establishing classical-quantum correspondence in quasiperiodic extended systems has proven to be difficult due to numerical limitations in approaching $\hbar\to 0$. All previous studies (see, e.g., [2]) addressing this question have been limited to $\hbar\approx 1$. Here we use the recently developed *renormalization group* (RG) approach [3] to study quantum transport for extremely small values of $\hbar$ up to $\approx 10^{-4}$.

The kicked Harper model [4,5,2] has emerged as an important model in quantum chaos literature. The system is given by the doubly periodic time-dependent Hamiltonian

$$H(t)=L\cos(p)+K\cos(q)\sum_{k=-\infty}^{\infty}\delta(t-k). \qquad (1)$$

Here q, p is a canonically conjugate pair of variables, usually considered on a cylinder $p\in(-\infty,\infty)$, $q\in[0,2\pi)$.

The classical dynamics of the kicked Harper model is determined by two parameters K and L. In the *asymmetric* case ($K\neq L$), the phase space, for small values of parameters, is stratified with KAM tori, which inhibit the transport on a global scale. For $L>K$, or $K>L$, these tori barriers limit the transport along p, or q, directions, respectively. As we show below, the two-dimensional (2D) parameter space exhibits intricately mixed nondiffusive (KAM regime) and diffusive regions corresponding to global stochasticity. In contrast, in the *symmetric case* $K=L$, there are no KAM barriers to global transport but a separatrix, and it is the breakup of the separatrix that results in global diffusion. For large values of the parameters, both the symmetric and the asymmetric model exhibits mostly diffusive behavior with the exception of narrow windows in parameter space where the AMs give rise to superdiffusive transport [6].

The quantized system that is periodically kicked is described by the quasienergy states of the one step time evolution operator, introducing an additional parameter $\hbar$ into the problem. However, one hopes to recover $\hbar$ independent behavior (as $\hbar\to 0$) in order to establish quantum signatures of classical behavior. It is a well established fact that the RG approach provides the most effective tool in distinguishing ballistic, diffusive and localized transport. Here we use recently developed [3] dimer decimation approach to study transport characteristics of the quasienergy states in the small $\hbar$ limit. The RG method is applied [3] to the momentum lattice ($p_m=\hbar m$) representation of the kicked model [7,8]

$$\sum_{r=-\infty}^{\infty}B_r^m u_{m+r}=0, \qquad (2)$$

where the coefficients B_r^m are

$$B_r^m=J_r(\bar{K})\sin[\bar{L}\cos(m\hbar)-\pi r/2-\omega/2]. \qquad (3)$$

Here ω is the quasienergy and $\bar{K}=K/(2\hbar)$ and $\bar{L}=L/(2\hbar)$ are *renormalized parameters*. The tight-binding model (TBM) [Eq. (2)] effectively contributes only few terms as Bessel's functions exhibit fast decay when $|r|>|\bar{K}|$. Therefore, the TBM describes a lattice model with a finite range of interaction denoted as b ($b\approx\bar{K}$). In the limit of small $\bar{K}$, $\bar{L}$, ω, the TBM reduces to the Harper equation with $\epsilon=\hbar\omega$ [9]. We will choose $\hbar$ to be an irrational number with a golden tail: $\hbar/(2\pi)=1/(n_h+\sigma)$, where $\sigma=(\sqrt{5}-1)/2)$ is fixed and $\hbar$ is varied by varying the integer n_h. This corresponds to studying system sizes N_n, $n=1,2,\ldots$ determined from the Fibonacci equation $N_{n+1}=N_n+N_{n-1}$, with $N_0=1$, N_1

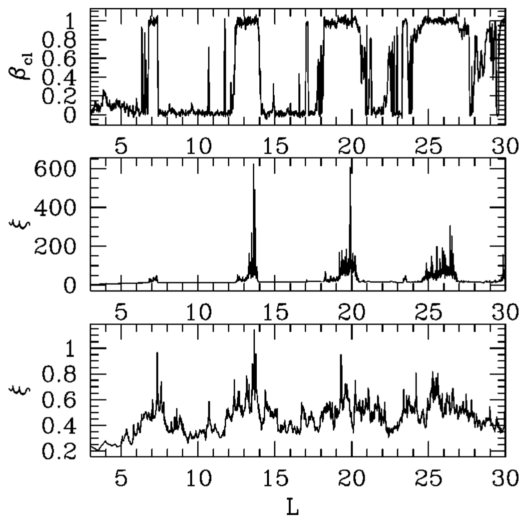

FIG. 1. For fixed $\bar{K}=0.4$, the figure describes a series of breakups of KAM barriers resulting in transitions to diffusive transport $\beta_{\rm cl}=1$ (top). Middle and the bottom figures show the corresponding plots of quantum localization length ξ for $n_h=200$ and $n_h=32$. The renormalization is carried out for system sizes increasing until the transmission probability becomes zero.

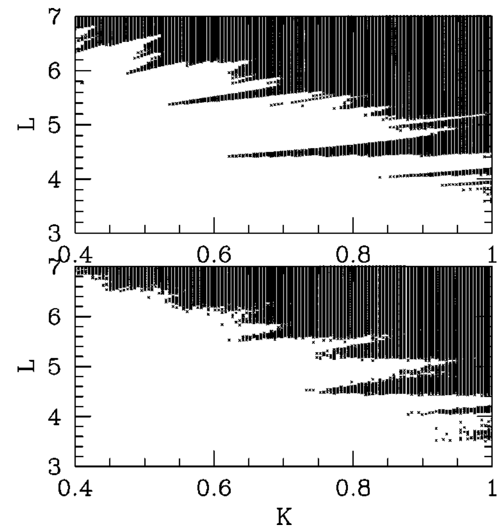

FIG. 2. The diffusive regime in 2D parameter space. The unshaded part (top) is in the regime of KAM barriers for momentum transport with $\beta_{\rm cl}=0$. The corresponding quantum plot (bottom) with $n_h=300$: In the unshaded part, the transport exponent is $\beta \leqslant -10$.

$=n_h$, corresponding to the nth successive rational approximant of the irrational number σ. The RG methods can be used to study system sizes up to 10^9 [10], which allows very large n_h and hence facilitates studying the semiclassical limit of the kicked model with a precision that has never been achieved before, to the best of our knowledge.

The transport characteristics of the quasienergy states are studied by computing the transmission probability T on the momentum lattice. This is achieved in two steps: first we decimate the lattice and then solve the scattering problem on the renormalized lattice [3]. The renormalization scheme makes the solution of the scattering problem for large lattices of size N very efficient, as the dimer decimation reduces the size of the *scattering region*. For a fixed $\hbar$, we compute the transmission probability $T(N)$ for various sizes N of the momentum lattice corresponding to a rational approximant of σ with denominator N. The scaling exponent $\beta =\lim_{N\to\infty} \ln T(N)/\ln N$ distinguishes extended, localized, and critical states respectively, described by $\beta(N)\to 0,\to -\infty$, and the *oscillatory function* $\beta(N_n)$ of n [3]. For the exponential localization, the quantity $\xi=-[\lim_{N\to\infty}(1/N)\ln T(N)]^{-1}$ has been found to be closely related to the localization length of the quasienergy eigenstate.

In contrast to the kicked rotor model, the kicked Harper model is found to exhibit a series of breakups and reformations of KAM barriers. These transitions, quantified by the exponent $\beta_{\rm cl}=\lim_{t\to\infty} \ln\langle[p(t)-p(0)]^2\rangle/\ln t$, are signaled by $\beta_{\rm cl}$ changing from 0 to ≈ 1. In the quantum model, our detailed analysis for various values of $\hbar$ and $L>K$ confirms the previously held view [7] that the quantum system remains localized in the classically diffusive regime. However, the classical transitions corresponding to diffusive transport manifest in a huge enhancement of the localization length. Results for an individual pure state $\omega=0$ are shown in Fig. 1. As demonstrated in the figure, a small $\hbar$ is crucial to see signatures of *all* classical transitions. We would like to point out that our results are consistent with the relation $\xi =1/2D/\hbar^2$ [8]. However, near the peaks (narrow windows in parameter space corresponding to the onset of classical transitions), the quantum transmission probability $T(N)$ ceases to look like a simple exponential $\sim\exp(-N/\xi)$ thus making quantitative comparison difficult.

An interesting aspect of the two-parameter kicked Harper model is that the boundary between the KAM and diffusive phases appears to be fractal as seen in Fig. 2. This behavior is reminiscent of the kicked rotor problem where the kicking potential consists of two harmonics [11]. Although somewhat smeared, the quantum model exhibits similar behavior: the boundary describes the transition to the enhancement of localization length ξ. It is remarkable that the quantum system feels the presence of all classical transitions and the fact that unlike the kicked rotor model, there is a whole hierarchy of transitions in the Harper model, which makes this model an important system in quantum chaos studies.

An important feature of kicked systems with toroidal phase space is the AMs that are regular (stable) space-time structures coexisting with the chaotic sea in phase space and are accompanied by a hierarchy of island chains inducing anomalous transport $\beta_{\rm cl}>1$. Figure 3 shows one such superdiffusive parameter window whose origin is traced to a period-8 AM. Once again, the quantum state $\omega=0$, although localized, exhibits a very strong enhancement of localization length in the classically superdiffusive regime. It should be noted that in contrast to the diffusive peaks, superdiffusive spikes are in fact groups of many spikes exhibiting sensitive dependence on the parameters and hence describe transport in fractal phase space.

It is rather surprising that a pure quantum state with $\omega =0$, which is an eigenstate for all parameter values, can ex-

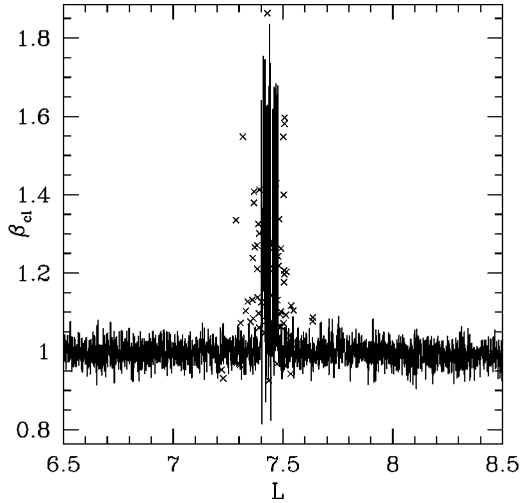

FIG. 3. For a fixed $K=0.86$ the figure shows classical anomalous transport due to AM. In the corresponding quantum results, the transmission probability was zero except at the crosses for $n_h=300$ and the lattice size $N=43\,289$. More precisely, the crosses indicate points where localization length varies between 400 to 800, which is about a tenfold increase from the localization length in the diffusive regime.

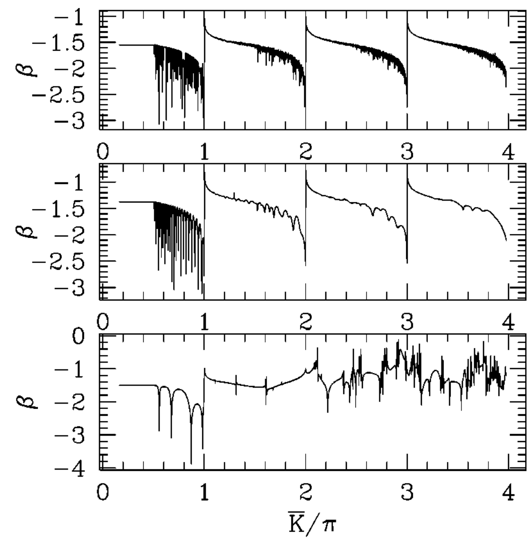

FIG. 4. Variations in β characterizing the transmission probability along the line $K=L$ in kicked Harper model for $n_h=10\,000$ (top), $n_h=250$ (middle), and $n_h=32$ (bottom). The plateau for $\bar{K}<\pi/2$ is the transport exponent for the Harper equation. Although the exponent β oscillates with N, all RG iterates show qualitatively the same behavior.

hibit such a clear signature of almost all the classical transitions. We would like to point out that we have only investigated the $L>K$ part of the parameter space whereas duality implies that analogous behavior will be seen for $K>L$ in q space.

In our earlier studies [3] for $\hbar \approx O(1)$, we found patches of ballistic (localized) regions for $L>K$ $(K>L)$ [3]. Numerical studies for smaller values of $\hbar$ suggest that the overall measure of the extended (localized) regimes for $L>K$ $(K>L)$ approaches zero as $\hbar \to 0$ [12].

We now discuss the symmetric Harper model with $K=L$. Here the quasienergy states remain critical and hence exhibit diffusive transport for all values of the kicking parameter K. As $\hbar \to 0$ (see Fig. 4), the transmission exponent becomes $\hbar$ independent provided we use the renormalized parameter $\bar{K}$ instead of the bare K. The model exhibits transmission characteristics of the Harper equation for $\bar{K} \leqslant \pi/2$. Precisely at $\bar{K}=\pi/2$, the transport exponent begins to exhibit an oscillatory behavior (with frequency proportional to $\hbar$). As opposed to classical mechanics, where infinitesimal perturbation leads to chaotic regions whose size increases as the perturbation increases, the perturbation of such a quantum system causes no immediate change in the transport characteristics. This suggests that the roots of these transitions may be topological [4]. It should be noted that the onset of oscillatory behavior is seen at higher odd multiples of $\pi/2$, however, the behavior becomes prominent only at very small $\hbar$. Another fascinating feature is that beyond $\bar{K}=\pi$, the transmission probability appears to be periodic in $\bar{K}$ with period π. Finally, the model exhibits a series of resonance-type transitions precisely at $\bar{K}=l\pi$, $l=1,2,\ldots$, characterized by a discontinuity in the transport exponent.

Recently, a semiclassical analysis [13] hinted at a possibility of a series of enhancements of transport at $\bar{K}=l/\pi/2$. The integer l is a kind of winding number that unfolds ω (confined to the interval $0\text{–}2\pi$). The fact that $\bar{K}$ and not K determines the thresholds for various transitions led to the conjecture that singular scaling of the parameters may be needed for classical-quantum correspondence. Furthermore, it was suggested that the semiclassical transitions at multiples of π are the quantum manifestations of the superdiffusive transitions due to period-1 and period-2 AMs, which respectively occur at even and odd multiples of π. However, the classical-quantum correspondence established in Figs. 1–3 rules out the necessity of any type of scaling of the parameters. It is possible that the transitions seen in the symmetric case are purely quantum mechanical in nature and may have their origin in resonances and/or topological changes. We should mention that a possibility of some sudden changes at multiples of $\pi/2$ also emerged in our analysis of the scattering problem. It turns out that the number of independent propagating solutions of the scattering problem [3], i.e., the dimension of the S matrix, increases by 1 at $\bar{K}_l=l\pi/2$, $l=1,2,\ldots$, matching with the discontinuities of the transport exponent $\beta(\bar{K})$.

The symmetric kicked Harper model is an interesting example of a nonintegrable system where the classical as well as the quantum transport is diffusive. It is in sharp contrast to the asymmetric case where the classically diffusive behavior corresponds to localized quantum transport. In view of this, it is rather surprising that in the asymmetric case the quantum system appears to respond to *all* the changes in the classical behavior, while in the symmetric case it remains insensitive to the variation in classical transport and instead repeats its behavior at every multiple of π. The fact that the transition thresholds are determined by $\bar{K}$ supports the view that these transitions are quantum mechanical in nature with no classical analog. Finally, semiclassical transport is $\hbar$ independent (meaning that it depends on $\hbar$ only through the rescaled parameter $\overline{K}$) in the symmetric case and preliminary studies show that for the asymmetric case, the $\hbar$ dependence exists only in the localization length while the fluctuations about the exponential envelope are again $\hbar$ independent. We hope that some of our results may be tested in experiments, as the symmetric kicked Harper model also describes a kicked oscillator model, which has an experimental realization in atom optics.

Inability of the quantum system to delocalize in the classically diffusive regime and mimic the classical behavior for arbitrary small values of $\hbar$ as confirmed by RG analysis, remains an open frontier. Earlier studies have suggested phase randomization [7] due to classical chaos as a mechanism for quantum dynamical localization. The fact that the kicked Harper model can exhibit localized, ballistic, and diffusive transport irrespective of the fact that the corresponding classical system is chaotic challenges the phase randomization as the underlying mechanism for localization. Here we would like to propose an alternative mechanism of dynamical localization: we speculate that the dynamical localization may be due to the cantori barriers. These are invariant quasiperiodic trajectories with infinite number of steps and provide an effective nonanalytic quasiperiodic potential. The possibility of localization in a quasiperiodic potential with *infinite steps* has been discussed recently [14]. We would like to emphasize that the scenario for localization due to cantori suggests that these barriers continue to inhibit transport even as $\hbar \rightarrow 0$ irrespectively of the flux through the holes in cantori. This scenario not only explains dynamical localization in the kicked rotor and Harper models (for $L > K$), but also accounts for ballistic transport in the kicked Harper model for $K > L$. Furthermore, we would like to attribute the features of the symmetric case to the absence of cantori, and conjecture that classically chaotic systems without KAM or cantori could exhibit intriguing features such as those shown in Fig. 4. We hope that further studies will put our speculative views on a solid footing.

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